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Indicators of Growth of Polynomials of Best Uniform Approximation to Holomorphic Functions on Compacta in C^N

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Let E be a compact and L-regular subset of C^N. Siciak has shown that a function f on E has a holomorphic extension to E_R—the interior of the level curve of the Siciak extremal function—if and only if $\limsup_{n\to\infty}(\sup_E |f-p_n|^{1/n}) \leqslant 1/R$ $(R>1)$, where p_n is a best approximating polynomial to f of degree not greater than n. The aim of this paper is to show that f has a holomorphic extension to E_R if for some sequence $\{p_n\}$ of the polynomials of best approximation to f

$$\limsup_{n\to\infty} \|\widehat{p_n}\|^{1/n} \leqslant (Rd(E))^{-1}$$

and if f has such an extension, for all $\{p_n\}$, there holds

$$\limsup_{n\to\infty} \|\widehat{p_n}\|^{1/n} \leqslant (Rc_m(E))^{-1}.$$

Here $\|\widehat{p_n}\|$ denotes a norm on the homogeneous terms of degree n in p_n and $c_m(E)$, $d(E)$ are some multidimensional counterparts of the logarithmic capacity and the Chebyshev constant, respectively.

0. INTRODUCTION

Let E be a compact set in the complex plane C, regular in the following sense: if G_E denotes the generalized Green function for the unbounded component D_∞ of the set $C\setminus E$ with a pole at the point $z=\infty$, then

$$\text{for all } \xi \in \partial D_\infty \qquad \lim_{z\to\xi} G_E(z) = 0.$$

For $R>1$, let $E_R := \{z\in C : G_E \leqslant \log R\} \cup C\setminus D_\infty$.

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Let f be a complex valued function defined on E and let $p_n(z) = a_n z^n + \cdots + a_0$ be its polynomial of best approximation in uniform norm $L^\infty(E)$, i.e.,

$$\|f - p_n\|_{L^\infty(E)} = \operatorname{dist}(f, P_n) := \inf\{\|f - q\|_{L^\infty(E)} : q \in P_n\},$$

where P_n denotes the set of all polynomials of degree not greater then n. One of the classical results of the constructive theory of functions is the following theorem due to S. N. Bernstein, if $E = [-1, 1]$ (see [1, p. 450]).

THEOREM 0.1 (cf. [2, Theorem 2.1; 16, Theorem 3]). *The function f has a holomorphic extension $\tilde{f}$ to the interior of E_R if and only if*

$$\lambda(f, E) := \limsup_{n \to \infty} |a_n|^{1/n} = (c(E)\, R)^{-1},$$

where $c(E)$ denotes the logarithmic capacity of the set E.

Thus, the number $\lambda(f, E)$ is an *indicator of growth of the polynomials of best uniform approximation to the function f* in a neighbourhood of the set E.

The aim of this paper is to introduce some similar indicators in the case of the space C^N (Definition 2.4).

Although the idea of the generalization given here seems to be natural, I have not seen it in available literature on the constructive theory of functions of several complex variables.

1. TYPE OF GROWTH OF THE LEJA–SICIAK EXTREMAL FUNCTION AND L-CAPACITY OF COMPACTA IN C^N

Let q be a norm in C^N. Denote

$$B_q(r) := \{z \in C^N : q(z) < r\}.$$

For every real-valued, non-negative function ϕ, defined on C^N, we define two numbers being indicators of growth of this function at infinity (cf. [8, Chap. 1]). Denoting

$$M_q(\phi, r) := \sup\{\phi(z) : z \in B_q(r)\}$$

we set

$$\rho_q(\phi) := \limsup_{r \to \infty} (\log M_q(\phi, r)/\log r)$$

and, if $\rho := \rho_q(\phi) < \infty$,

$$\sigma_q(\phi) := \limsup_{r \to \infty} (r^{-\rho} M_q(\phi, r)).$$

Since all norms in C^N are equivalent, the numerical value of $\rho_q(\phi)$ does not depend on the choice of a norm in C^N, while for every two norms p and q the numbers $\sigma_q(\phi)$ and $\sigma_p(\phi)$ may be different but have the same character, i.e., either $\sigma_q(\phi) = \sigma_p(\phi) \in \{0, \infty\}$ or they are finite positive numbers (see [8, Chap. 1]).

DEFINITION 1.1. The numbers $\rho(\phi)$ and $\sigma_q(\phi)$ are called *order* and *type of the function* ϕ, respectively, with respect to the chosen norm q.

Zakharyuta [18] and Siciak [13] proposed the following generalization of the logarithmic capacity of a compact set E in C^N.

Denote by P_n the set of all polynomials in N complex variables of degree not greater than n ($n = 0, 1, 2, 3, ...$).

Let, for $z \in C^N$,

$$\Phi_E(z) := \sup\{|p(z)|^{1/n} : p \in P_n, n \in N, \|p\|_{L^\infty(E)} \leqslant 1\}$$

denote *Siciak's extremal function of the set* E (cf. [12]). In the case of $N = 1$ the function Φ_E is equal to Leja's extremal function L_E that has the property

$$\log L_E(z) = G_E(z) \qquad \text{for} \quad z \in D_\infty$$

(cf. [7, p. 274]).

DEFINITION 1.2. For every compact set E in C^N the number

$$c_q(E) := \liminf_{r \to \infty} (r/M_q(\Phi_E, r))$$

is called the *L-capacity* of the set E with respect to the norm q.

Observation 1.3. The L-capacity $c_q(E)$ of the set E is the inverse of the type $\sigma_q(\Phi_E)$ of the extremal function Φ_E.

As a corollary to [13, Theorem 3.10 and Corollary 3.9], we get the following characterization of *pluripolar sets* in C^N, i.e., such sets $E \subset C^N$ that $E \subset \{z \in C^N : u(z) = -\infty\}$, where u is a plurisubharmonic function not identically equal to $-\infty$.

PROPOSITION 1.4. *Let q be a norm and E be a compact set in C^N. Then*

(1) *The set E is not pluripolar if and only if Φ_E is a function of order* 1 *and of normal type, i.e.,* $\sigma_q(\Phi_E) \in (0, \infty)$.

(2) *If E is a pluripolar set, then the order of the function Φ_E is infinite.*

The following relation between L-capacity of a compact set E and the set $E_R := \{z \in C^N : \Phi_E(z) \leqslant R\}$ for $R > 1$ is furnished by Mazurek's lemma (cf. [13, Proposition 5.11]).

PROPOSITION 1.5. *For every compact set in C^N and every number $R > 1$ the following equalities hold*

$$c_q(E_R) = Rc_q(E)$$

for every norm q in C^N, and

$$\sigma_q(\Phi_{E_R}) = R^{-1}\sigma_q(\Phi_E),$$

if E is not pluripolar.

2. NECESSARY CONDITIONS FOR ANALYTICITY OF FUNCTIONS IN A NEIGHBOURHOOD OF A COMPACT SET IN C^N

Denote by H_n the subset of P_n containing all homogeneous polynomials of degree n. Let m be *the polydisc norm* in C^N, i.e.,

$$m(z) := \max\{|z_i|, i \in \{1, ..., N\}\}, \qquad z = (z_1, ..., z_N) \in C^N.$$

Denote by $D_N(r)$ the polydisc centered at zero and of radii equal to r. With the previous notations, we have $D_N(r) = B_m(r)$.

Let $p_n \in P_n$, so that $p_n(z) = \sum_{|\alpha| \leqslant n} a_\alpha z^\alpha$, where α is a multi-index from $N_0^N := \{\eta = (\eta_1, \eta_2, ..., \eta_N) : \eta_i = 0, 1, 2, ...$ for $i = 1, 2, ..., N\}$ and $|\alpha|$ denotes its length, i.e., $|\alpha| := \alpha_1 + \cdots + \alpha_N$.

Notation 2.1. By $\widehat{p_n} \in H_n$ we denote the homogeneous polynomial $\sum_{|\alpha| = n} a_\alpha z^\alpha$ corresponding to a polynomial $p_n(z) = \sum_{|\alpha| \leqslant n} a_\alpha z^\alpha \in P_n$.

With the above notation we have

LEMMA 2.2. *For every non-pluripolar set E in C^N and every $p \in P_n$ the following estimate holds*

$$\|\widehat{p_n}\|_{D_N(1)} \leqslant (\sigma_m(\Phi_E))^n \|p_n\|_E.$$

Proof. By the definition of the extremal function we have

$$\|p_n\|_{D_N(R)} \leqslant (\|\Phi_E\|_{D_N(R)})^n \|p_n\|_E, \qquad R > 1,$$

which is a generalized Bernstein–Walsh inequality. Since $\|R^{-n}(p_n - \widehat{p_n})\|_{D_N(R)} \to 0$, when $R \to \infty$ and $\|R^{-n}\widehat{p_n}\|_{D_N(R)} = \|\widehat{p_n}\|_{D_N(1)}$, by Definition 1.1 we obtain the assertion of the lemma.

Remark 2.3. In the case of $N=1$ the lemma was proved in [2] (cf. Lemma 4.1).

Let E be *an L-regular* compact set in C^N (i.e., such a set that Siciak's extremal function Φ_E is continuous in C^N) and let f be a holomorphic function in the set E_R, for a number $R>1$. Denote by $\{p_n\}_{n\in N}$ any sequence of polynomials of best uniform approximation of f, i.e.,

$$\|f - p_n\|_E = \inf\{\|q - f\|_E : q \in P_n\}.$$

Since, for $N \geq 2$ the set P_n is not in general *a Haar subspace* of the space $C(E)$ of all continuous functions defined on the set E (considered with the $L^\infty(E)$-norm), the polynomial p_n is not necessarily unique (see Example 2.5).

Let $P(f, E)$ denote the family of all sequences of polynomials of best approximation to the function f in $L^\infty(E)$-norm.

For every $\{p_n\} \in P(f, E)$ we set

$$\lambda(f, E, \{p_n\}) := \limsup_{n\to\infty} (\|\widehat{p_n}\|_{D_N(1)})^{1/n}.$$

DEFINITION 2.4. The number

$$\lambda(f, E) := \sup\{\lambda(f, E, \{p_n\}) : \{p_n\} \in P(f, E)\}$$

will be called the *indicator of growth of the polynomials of best approximation to the function f on the set E.*

There arises a question of whether the number $\lambda(f, E, \{p_n\})$ really depends on the choice of the sequence $\{p_n\}$ from $P(f, E)$. The purpose of Example 2.5 is to answer this question.

EXAMPLE 2.5. Following the one-dimensional idea of Chebyshev and using the Kolmogorov test one can find the polynomials of best approximation to the function

$$f(z, w) := \frac{1}{(z-a)(w-b)}, \qquad |a| \geq |b| > 1 \tag{2.1}$$

on the unit bidisc $D_2(1) = \{(z, w) \in C^2 : |z| \leq 1, |w| \leq 1\}$ (see [10, Chap. 4.3]).

The set of best approximating polynomials of degree n to f in $L^2(D_2(1))$-norm contains all polynomials of the form

$$p_{n,k}(z,w) := q_{n-k}(z)\, r_k(w), \qquad k \in \{0, 1, 2, \ldots, n\}, \tag{2.2}$$

where

$$q_l(z) = \frac{1}{z-a} + \frac{1}{a^l(|a|^2-1)} \frac{1-\bar{a}z}{z-a} z^l$$
$$= -\frac{\bar{a}}{a^l(|a|^2-1)} z^l - a^{-l} \frac{z^l - a^l}{z-a}$$

and

$$r_m(w) = -\frac{\bar{b}}{b^m(|b|^2-1)} w^m - b^{-m} \frac{w^m - b^m}{w-b}.$$

Hence

$$\widehat{p_{n,k}}(z,w) = \bar{a}a^{-n+k}(|a|^2-1)^{-1}\bar{b}b^{-k}(|b|^2-1)^{-1} z^{n-k} w^k$$

and

$$\|\widehat{p_{n,k}}(z,w)\|_{D_2(1)} = |a|^{-n+k+1}(|a|^2-1)^{-1}|b|^{-k+1}(|b|^2-1)^{-1}.$$

It is easily seen that

$$|a|^{-1} \leqslant \lambda(f, D_2(1), \{p_{n,k}\}) \leqslant |b|^{-1}$$

and for each point ρ from the closed interval $[|a|^{-1}, |b|^{-1}]$ there is a subsequence from $P(f, D_2(1))$ of the form (2.2), which converges to ρ.

This gives an example of non-uniqueness of the polynomials of best approximation and shows that the value of $\lambda(f, D_2(1))$ depends indeed on the choice of $\{p_n\} \in P(f, D_2(1))$.

With the previous notations, the following theorem is true.

THEOREM 2.6. *If E is an L-regular compact set in C^N and f is a holomorphic function in the interior of E_R, for a number $R > 1$, then*

$$\lambda(f, E) \leqslant R^{-1}\sigma_m(\Phi_E) = (Rc_m(E))^{-1}.$$

Proof. By the Bernstein–Walsh–Siciak theorem (see [12, Theorem 10.1]), for every number $r \in (1, R)$, any sequence of polynomials p_n from $P(f, E)$ tends to f uniformly on E_r, so the upper bound

$$M_r := \sup\{\|p_n\|_{E_r} : n \in N\}$$

is finite. By Lemma 2.2, Proposition 1.4, and the definition of $\lambda(f, E, \{p_n\})$

$$\lambda(f, E, \{p_n\}) \leqslant r^{-1}\sigma_m(\Phi_E).$$

By the arbitrariness of the choice of $r \in (1, R)$ and $\{p_n\} \in P(f, E)$, we conclude that the theorem holds.

EXAMPLE 2.7. Let f be the function defined in Example 2.5. Regarding Definitions 1.2 and 2.4 we obtain $c_m(D_2(1)) = 1$ and $\lambda(f, D_2(1)) = |b|^{-1}$. It is easily seen that f admits an analytic extension to the interior of $D_2(|b|)$, which corresponds to $\{(z, w) \in C^2 : \Phi_{D_2(1)}(z, w) \leqslant |b|\}$.

This illustrates the previous theorem.

3. Some Constants of Chebyshev Type

Take a multi-index $\alpha \in N_0^N$ and choose a polynomial l_α in the set $P_{|\alpha|-1}$ such that

$$\inf\{\|z^\alpha + p\|_E, p \in P_{|\alpha|-1}\} = \|t_\alpha\|_E, \tag{3.1}$$

where $t_\alpha = z^\alpha + l_\alpha$.

Remark 3.1. In the case of $N = 1$ and $E = [-1, 1]$, the polynomials t_α were introduced by P. L. Chebyshev (see [4, p. 195; 5, Vol. III, pp. 24–48]). Zériahi [19] has investigated such polynomials in the case of the norm $L^2(E)$ and called them *extremal polynomials of the set E in* C^N.

By the definition of t_α, for every $\alpha, \beta \in N_0^N$ the following inequality holds

$$\|t_{\alpha+\beta}\|_E \leqslant \|t_\alpha\|_E \, \|t_\beta\|_E. \tag{3.2}$$

Let us consider the number

$$d_n(E) = \sup\{\|t_\alpha\|_E : \alpha \in N_0^N, |\alpha| = n\}. \tag{3.3}$$

We will show that for all natural numbers k, l we have the inequality

$$d_{k+l}(E) \leqslant d_k(E)\, d_l(E). \tag{3.4}$$

Fix $\alpha \in N_0^N$ such that $|\alpha| = k + l$ and $d_{k+l}(E) = \|t_\alpha\|_E$. For every number $j \in \{0, 1, ..., k+l\}$ choose the subset A_j of the set N_0^N as

$$A_j := \{\beta \in N_0^N : |\beta| = j \text{ and } \exists \gamma \in N_0^N : \beta + \gamma = \alpha\}.$$

By inequality (3.2) it follows that

$$d_{k+l}(E) = \|t_\alpha\|_E \leqslant \sup_{\beta \in A_k} \|t_\beta\|_E \sup_{\gamma \in A_l} \|t_\gamma\|_E \leqslant d_k(E)\, d_l(E)$$

which proves the inequality (3.4) and the existence of the limit

$$d(E) = \lim_{n \to \infty} (d_n(E))^{1/n}$$

(cf. [7, p. 257]).

DEFINITION 3.2. We call the number $d(E)$ the *Chebyshev constant of the compact set* E in C^N.

Let Γ be the set of all bijections $\kappa: N \to N_0^N$ such that $|\kappa(j)| \leqslant |\kappa(j+1)|$, for every $j \in N$. Zakharyuta [17] (see also [6]) has introduced the following two constants for the compact set E:

$$\tau_+(E, \kappa) := \limsup_{j \to \infty} (D_\kappa(j))^{1/|\kappa(j)|} \tag{3.5}$$

and

$$\tau_-(E, \kappa) := \liminf_{j \to \infty} (D_\kappa(j))^{1/|\kappa(j)|}, \tag{3.6}$$

where $\alpha \in N_0^N$ and

$$D_\kappa(j) := \inf\{\|z^{\kappa(j)} + k\|_E, k \in \Pi_\kappa(j)\},$$

$$\Pi_\kappa(j) := \left\{\sum_{i=1}^{j-1} c_i z^{\kappa(i)} : c_i \in C\right\}.$$

Denote by

$$\tau_+(E) := \sup\{\tau_+(E, \kappa), \kappa \in \Gamma\},$$

$$\tau_-(E) := \inf\{\tau_-(E, \kappa), \kappa \in \Gamma\}.$$

One can easily see that

$$d(E) \geqslant \tau_+(E, \kappa), \qquad \text{for all} \quad \kappa \in \Gamma. \tag{3.7}$$

In particular $d(E) \geqslant \tau_+(E) \geqslant \tau_-(E)$.

There arises the question of whether there exists a bijection $\kappa \in \Gamma$ such that the equality in (3.7) holds. We are going to define *an extremal bijection* κ_E, associated with *the extremal polynomials* t_α (cf. Remark 3.1), whose properties allow us to answer this question in the affirmative.

From the set $M_n := \{\alpha \in N_0^N : |\alpha| = n\}$ of the multi-indices of length n choose a multi-index ζ_n such that $d_n(E) = \|t_{\zeta_n}\|_E$, and next order the set

M_n in such a manner that ζ_n precedes all remaining elements of M_n. Supposing, moreover, that every multi-index $\beta \in M_{n-1}$ precedes every multi-index $\gamma \in M_n$, we define the required bijection $\kappa_E \in \Gamma$ so that the order given by it is identical with the one prescribed above. In particular, we have

$$\kappa_E^{-1}(\beta) < \kappa_E^{-1}(\zeta_n) \qquad \text{implies} \quad |\beta| \leqslant n-1, \text{ for each } \beta \in N_0^N.$$

One can easily see that

$$\Pi_{\kappa_E}(\kappa_E^{-1}(\zeta_n)) = P_{n-1},$$

hence

$$D_{\kappa_E}(\zeta_n) = \inf\{\|z^{\zeta_n} + k\|_E, k \in P_{n-1}\} = \|t_{\zeta_n}\|_E = d_n(E),$$

and we obtain the following

COROLLARY 3.3. *For every compact set* $E \subset C^N$

$$d(E) = \tau_+(E) \geqslant \tau_+(E, \kappa) \geqslant \tau_-(E), \qquad \kappa \in \Gamma.$$

EXAMPLE 3.4. The extremal bijection defined above depends on the set E. Let $E := \{(z_1, z_2) : |z_1| \leqslant 1, z_2 = 0\}$ and $F := \{(z_1, z_2) : z_1 = 0, |z_2| \leqslant 1\}$. It is easy to check that one can define κ_E and κ_F as

$\kappa_E(1) = (0, 0)$,

$\kappa_E(2) = (1, 0)$, $\quad \kappa_E(3) = (0, 1)$,

$\kappa_E(4) = (2, 0)$, $\quad \kappa_E(5) = (1, 1)$, $\quad \kappa_E(6) = (0, 2)$,

$\kappa_E(7) = (3, 0)$, $\quad \kappa_E(8) = (2, 1)$, $\quad \kappa_E(9) = (1, 2)$, $\quad \kappa_E(10) = (0, 3)$,

...

and

$\kappa_F(1) = (0, 0)$,

$\kappa_F(2) = (0, 1)$, $\quad \kappa_F(3) = (1, 0)$,

$\kappa_F(4) = (0, 2)$, $\quad \kappa_F(5) = (2, 0)$, $\quad \kappa_F(6) = (1, 1)$,

$\kappa_F(7) = (0, 3)$, $\quad \kappa_F(8) = (3, 0)$, $\quad \kappa_F(9) = (2, 1)$, $\quad \kappa_F(10) = (1, 2)$,

...

Moreover, neither κ_E is an extremal bijection for the set F nor κ_F for E.

Remark 3.5. In the case of $N=1$, the set Γ has only one element, hence by (3.4) the limits in (3.5), (3.6) exist and are equal to the transfinite diameter of the set E as well as its logarithmic capacity $c(E)$ (see, e.g., [15, Theorem III.26]).

For $N>1$, the following example due to Zakharyuta shows that, in general, $\tau_+(E)>\tau_-(E)$.

EXAMPLE 3.6 (cf. [17]). Consider $E:=\{z\in C^2 : |z_1|\leqslant 1, z_2=0\}$. Then $\tau_+(E)=1$, but $\tau_-(E)=0$. Moreover, this illustrates the fact that the pluripolarity of E need not imply $\tau_+(E)=0$.

Hence, by [17, Sect. 5, Theorem 1 and Sect. 7, Corollary 6] and Corollary 3.3 we obtain the following relations between the constant $d(E)$, Zakharyuta's constants of the Chebyshev type, and the L-capacity $c_m(E)$ of the set E, associated with the polydisc norm (cf. Definition 1.2).

COROLLARY 3.7. *For every compact set E in C^N we have*

$$d(E)=\tau_+(E,\kappa_E)=\tau_+(E)\geqslant c_m(E).$$

Moreover, from Proposition 1.4, we derive the following

COROLLARY 3.8. *For every non-pluripolar compact set E in C^N the constant $d(E)$ is positive.*

Similar relations among capacities in C^N one can find also in [9].

4. Sufficient Conditions for Holomorphic Extension of Functions to a Neighbourhood of a Compact Set in C^N

Denote by $W(E)$ the closure in the $L^\infty(E)$-norm of the algebra of all polynomials in N complex variables, where E is a compact set in C^N. Fix a function f in $W(E)$ and consider a sequence of polynomials $\{p_n\}_{n\in N}$ of best approximation to the function f in the $L^\infty(E)$-norm. Observe that, writing the polynomial p_{n+1} in the form

$$p_{n+1}(z)=\sum_{|\alpha|\leqslant n+1} a_\alpha^{(n+1)}z^\alpha,$$

we get

$$\|f-p_n\|_E\leqslant\left\|f-\left(p_{n+1}-\sum_{|\alpha|=n+1} a_\alpha^{(n+1)}t_\alpha\right)\right\|_E,$$

where t_α is a polynomial defined by (3.1), with $|\alpha| = n+1$. Hence

$$\|f - p_n\|_E \leqslant \|f - p_{n+1}\|_E + d_{n+1}(E) \sum_{|\alpha| = n+1} |a_\alpha^{(n+1)}|. \tag{4.1}$$

LEMMA 4.1. *With the above notations, the following equalities hold*

$$\limsup_{n \to \infty} \Big(\sum_{|\alpha| = n} |a_\alpha^{(n)}| \Big)^{1/n} = \limsup_{n \to \infty} \big(\max_{|\alpha| = n} |a_\alpha^{(n)}|\big)^{1/n} = \limsup_{n \to \infty} \big(\|\widehat{p_n}\|_{D_N(1)}\big)^{1/n}.$$

It is easily seen that

$$\max_{|\alpha| = n} |a_\alpha^{(n)}| \leqslant \|\widehat{p_n}\|_{D_N(1)} \leqslant h_{n,N} \max_{|\alpha| = n} |a_\alpha^{(n)}|,$$

where $h_{n,N}$ denotes the number of N-indices of length n. Since $h_{n,N} = \binom{n+N-1}{N-1}$, we have $\lim_{n \to \infty} (h_{n,N})^{1/n} = 1$, and the above equalities hold.

THEOREM 4.2. *Let E be an L-regular compact set in C^N and f be a function from $W(E)$. A sufficient condition for the existence of a holomorphic extension $\tilde{f}$ of the function f to the interior of the set E_R, for a certain number $R > 1$, is that*

$$\lambda(f, E, \{p_n\}) \leqslant (Rd(E))^{-1}, \tag{4.2}$$

where $\{p_n\}$ is a sequence of polynomials from $P(f, E)$.

Proof. By Corollary 3.8, if E is an L-regular set, then $d(E) > 0$. Fix a number $r \in (1, R)$. By Lemma 4.1 there exists a number $n_r \in N$ such that

$$\sum_{|\alpha| = n} |a_\alpha^{(n)}| \leqslant (d(E)\, r)^{-n},$$

for every natural number $n \geqslant n_r$. Fix a positive number ε so that $r_\varepsilon := d(E) r / [d(E) + \varepsilon] > 1$. By the definition of $d(E)$, there exists a number $n_\varepsilon \geqslant n_r$ such that

$$d_n(E) \leqslant (d(E) + \varepsilon)^n, \qquad \text{for} \quad n \geqslant n_\varepsilon. \tag{4.3}$$

Then by (4.1) we obtain

$$\|f - p_n\|_E \leqslant \|f - p_{n+1}\|_E + (r_\varepsilon)^{-(n+1)}, \qquad \text{for} \quad n \geqslant n_\varepsilon.$$

Since $f \in W(E)$, for every number $n \geqslant n_\varepsilon$ we can find a number $k_n \in N$ so large that $\|f - p_{n+k_n}\|_E \leqslant (r_\varepsilon)^{-n}$. Repeating the above argument k_n times we obtain

$$\|f - p_n\|_E \leqslant M(\varepsilon)\, r_\varepsilon^{-n} \qquad \text{for} \quad n \geqslant n_\varepsilon,$$

where $M(\varepsilon)$ is a constant not depending on n. Hence, since $\varepsilon > 0$ and $r \in (1, R)$ have been chosen arbitrarily, it follows that

$$\limsup_{n \to \infty} \|f - p_n\|_E^{1/n} \leqslant R^{-1}.$$

Consequently, by the Bernstein–Walsh–Siciak theorem (cf. [12, Theorem 10.1]) it follows that there exists a holomorphic extension $\tilde{f}$ of the function f to the set E_R.

5. The One-Dimensional Case

In 1929 S. N. Bernstein (cf. [1, p. 450]) showed that *if the polynomials of best approximation in the norm* $L^2([-1, 1], \mu)$ *with* $d\mu(x) := (1-x)^\alpha (1+x)^\beta \, dx$, *to a positive function* f *defined on the interval* $[-1, 1]$, have no zeroes in the interior of the ellipse $E_R := \{z \in C : |z + \sqrt{z^2 - 1}| \leqslant R\}$, $R > 1$, then f has a holomorphic extension to the interior of E_R.

This result was generalized by Pleśniak [11] in the case when E is a compact set in C and μ a measure on E such that the pair (E, μ) satisfies *the Leja polynomial condition* (cf. [7, p. 273]).

For the uniform norm case on the interval $[-1, 1]$, Bernstein's theorem was proved by Borwein [3]. Blatt and Saff [2] and independently Wójcik [16] generalized Borwein's result as follows.

Theorem 5.1. *Let E be an L-regular compact subset of the complex plane C and f be a function from $W(E)$. Denote by $p_n = a_n z^n + \cdots + a_0$ the nearest polynomial to f from the set P_n with respect to the $L^\infty(E)$-norm. Let R be a number greater than* 1.

Then the following conditions are equivalent.

(1) *There exists a holomorphic extension of the function f to the interior of the set E_R;*

(2) *For every number $r \in (1, R)$, there exists a number $A_r \in C$ such that $p_n(z) - A_r \neq 0$ for every $z \in E_r$ and all $n \in N$;*

(3) $\limsup_{n \to \infty} |a_n|^{1/n} \leqslant (Rd(E))^{-1}$.

It is easy to see that Theorems 2.6 and 4.2 extend the equivalence of conditions (1) and (3) of Theorem 5.1 to the case of several complex variables which seems to be a first step for proving the implication (2) ⇒ (1) in the case of $N > 1$. Simkani [14] has proved this in the case when E is the unit polydisc in C^N. This problem, posed many years ago by Pleśniak, remains still open in the general case.

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